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RESEARCH MEMORANDUM

APPLICATION OF WING-BODY THEORY TO DRAG REDUCTION
AT LOW SUPERSONIC SPEEDS

By Barrett S. Baldwin, Jr., and Robert R. Dickey

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**NATIONAL ADVISORY COMMITTEE
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Page 10, lines 11 and 14: Replace the expression $\left[\frac{q}{2} - 1 \right]$ by q .

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SUMMARY

A method is developed for extending to higher Mach numbers the region of low drag attainable for wing-body combinations by the use of the transonic area rule. It is found that to a good approximation, the drag depends only upon the longitudinal distributions of area and moments of area about the vertical plane of symmetry parallel to the free-stream direction. The essential requirement of the method is that the longitudinal development of the moments of area be smooth and gradual.

Results of an experimental investigation conducted in the Ames 2- by 2-foot transonic wind tunnel to test the theory are presented. The results in essence confirm the predictions of the theory in that the zero-lift wave drag of a wing-body configuration over a range of low supersonic Mach numbers as well as at sonic speed is reduced when auxiliary bodies are mounted on the wing.

INTRODUCTION

R. T. Jones has expressed the theory of wing-body wave drag at supersonic speeds in a form which illustrates the dependence of the drag upon the longitudinal distributions of the cross-sectional areas of the complete configuration intercepted by planes inclined at the Mach angle of the flow (see ref. 1). The derivation contains as a special case for a Mach number of one the transonic area rule introduced by Whitcomb (ref. 2) wherein the intercepting planes are normal to the longitudinal axis of the configuration. It was concluded in reference 1 that the modification of a wing-body combination in accordance with the transonic area rule would generally be expected to result in drag reductions at near sonic speeds; however, it was pointed out that, at higher supersonic Mach numbers, this modification would sometimes result in drags greater than that of the original configuration. In reference 1 a method for contouring the fuselage of a wing-body combination was presented which achieved drag reductions at particular

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supersonic design Mach numbers, but only at the expense of increasing the sonic-speed drag compared with that of the corresponding transonic-area-rule configuration. Thus, it appears that the methods for minimizing wave drag described in references 1 and 2, which are both based on the longitudinal distributions of cross-sectional area, are effective for only a limited Mach number range.

In the present paper, Jones' generalized zero-lift wave-drag formula is re-examined in an attempt to develop a method for minimizing the wave drag of a wing-body combination over a wider range of Mach numbers.

ANALYSIS

Calculation of Zero-Lift Wave Drag

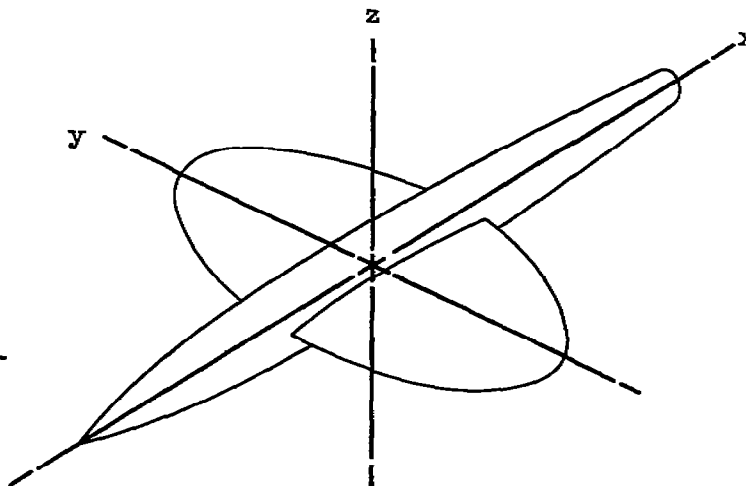
It has been pointed out in reference 1 that the transonic area rule was predicted by the linear theory, but was discounted because basic assumptions of the theory are violated in this application. It has been suggested by Jones that other predictions of the linear theory which may have been overlooked should be systematized and investigated experimentally. Thus, in the present analysis a possibly unwarranted emphasis is placed on the formal predictions of the linear theory at Mach numbers near one.

In reference 3 methods are given for calculating the aerodynamic forces on airplane configurations utilizing very few assumptions other than those needed for linearization. An additional approximation is employed in reference 1 to relate the supersonic zero-lift wave drag of a wing-body combination to the drags of a series of equivalent bodies of revolution each of which is determined from the cross-sectional areas intercepted on the configuration by a set of parallel Mach planes. The result of reference 1 coincides with the more exact result of reference 3 at sonic speed, and the deviation with increasing Mach number is expected to be small in a limited range of Mach numbers as long as the configuration is a conventional monoplane type.

In the interest of obtaining a result in terms of familiar geometric concepts and to facilitate calculations, the method of reference 1, termed the "Mach plane method," will be employed here. This approximate theory greatly simplifies the discussion of gross effects of variations in the design of wing-body combinations.

As a preface to the development of the method of the present paper for calculating drag, the Mach plane method will be briefly reviewed. Symbols are defined in Appendix A.

Mach plane method.— Consider a wing-body combination such as shown in sketch (a). Let x be the coordinate in the free-stream direction, y the spanwise coordinate, and z the remaining Cartesian coordinate in the thickness direction, with the origin at the center of the body.



Sketch (a)

A Mach plane can be defined as a plane with its normal at an angle of $\tan^{-1}(1/\beta)$ to the x axis. Let (x', β, ϕ) denote the Mach plane which intersects the x axis at x' and has the projection of its normal on the yz plane at an angle ϕ to the y axis. Let $S(x', \beta, \phi)$ be the area of the projection on the yz plane of the cross-sectional area intercepted on the configuration by the Mach plane (x', β, ϕ) . Then the drag of the configuration is the average with respect to ϕ of the drags of the equivalent bodies of revolution defined by the area distributions $S(x', \beta, \phi)$.

A method introduced in reference 4 is used in reference 1 to evaluate the drag of each equivalent body of revolution. The variable θ is defined by the relation

$$x' = \frac{l}{2} \cos \theta \quad (1)$$

where l is the length of the equivalent body. Then a set of quantities $A_n(\beta, \phi)$ are defined as the coefficients of $\sin n\theta$ in a Fourier series expansion of $\frac{\partial S(x', \beta, \phi)}{\partial x'}$. Consequently the $A_n(\beta, \phi)$ can be determined from the relation

$$A_n(\beta, \phi) = \frac{2}{\pi} \int_{-\pi}^0 \frac{\partial S(x', \beta, \phi)}{\partial x'} \sin(n\theta) d\theta \quad (2)$$

Finally, the drag of the configuration is given by

$$D = \frac{\rho V^2}{16} \int_0^{2\pi} \sum_{n=1}^{\infty} n [A_n(\beta, \phi)]^2 d\phi \quad (3)$$

Within the framework of the linear theory this result is valid only for equivalent bodies of revolution with no discontinuities in the gradients of the area distributions.

It should be noted that unless all parts of the configuration lie between the nose Mach cone and the forward Mach cone from the tail, the equivalent body length, l , will be greater than the actual body length in some cases. However, by consideration of streamwise body extensions of vanishingly small cross-sectional area, it can be seen that a constant value of l equal to or greater than the length of the longest equivalent body can be used in equation (1).

Series-expansion method.— In this section the Fourier series coefficients defined in equation (2) will each be expanded in a finite series so that the drag formula can be expressed as a power series in powers of β . This manipulation leads to an expression of the drag in terms of a convenient set of geometric parameters which were not apparent in the Mach plane method.

By the use of equation (1), equation (2) can be written as

$$A_n(\beta, \phi) = - \frac{2}{l} \frac{2}{\pi} \int_{-l/2}^{l/2} \frac{\partial S(x', \beta, \phi)}{\partial x'} \frac{\sin(n\theta)}{\sin \theta} dx'$$

or after a partial integration

$$A_n(\beta, \phi) = \left(\frac{2}{l} \right)^2 \frac{2}{\pi} \int_{-l/2}^{l/2} S(x', \beta, \phi) \frac{d}{d \left(\frac{x'}{l/2} \right)} \frac{\sin n\theta}{\sin \theta} dx' \quad (4)$$

provided that $\frac{\partial S(x', \beta, \phi)}{\partial x'}$ and $S(x', \beta, \phi)$ are zero at the nose and tail.¹

¹For a practical configuration where the area distribution is not zero at the tail, the distribution of a Kármán ogive having a base area and length equal to that of the configuration under consideration can be subtracted from $S(x', \beta, \phi)$ so that the resulting equivalent-body area distributions will be zero at the nose and tail. The drag due to the part removed can then be calculated by means of equation (2) rather than equation (4). The choice of a Kármán area distribution insures that there will be no interaction drag from the part removed as long as all configuration parts lie within the Mach cone from the body nose and within the forward Mach cone from the tail, as can be seen by the use of equations (2) and (3).

The second factor of the integrand of equation (4) can be expanded in a finite series of powers of $\frac{x'}{l/2}$ given by

$$\frac{d}{d\left(\frac{x'}{l/2}\right)} \frac{\sin n\theta}{\sin \theta} = \sum_{m=0}^{n-2} b_{nm} \left(\frac{x'}{l/2}\right)^m, \quad n \geq 2 \quad (5)$$

where

$$b_{nm} = \begin{cases} (-1)^{\left(\frac{n-m-2}{2}\right)} \frac{\left(\frac{n+m}{2}\right)! 2^{m+1}}{\left(\frac{n-m-2}{2}\right)! m!} & \text{for even values} \\ & \text{of } n - m \\ \text{zero for odd values of } n - m \end{cases} \quad (6)$$

Substituting equation (5) into (4) and interchanging the order of summation and integration yields

$$A_n(\beta, \varphi) = \sum_{m=0}^{n-2} b_{nm} \left(\frac{2}{l}\right)^2 \frac{2}{\pi} \int_{-l/2}^{l/2} S(x', \beta, \varphi) \left(\frac{x'}{l/2}\right)^m dx', \quad n \geq 2 \quad (7)$$

At this point, a more explicit expression for $S(x', \beta, \varphi)$ in terms of the configuration geometry is needed for substitution in equation (7). Let $t(x, y)$ be the thickness distribution of the configuration including that of the body or bodies. It will be assumed that the distance of all parts from the xy plane is small enough that $S(x', \beta, \varphi)$ can be approximated by

$$S(x', \beta, \varphi) = \int_{y_1(x', \beta, \varphi)}^{y_2(x', \beta, \varphi)} t(x' + \beta y \cos \varphi, y) dy \quad (8)$$

where $y_1(x', \beta, \varphi)$ and $y_2(x', \beta, \varphi)$ define the edges of the configuration intercepted by the Mach plane (x', β, φ) . Equation (8) represents a planar approximation. Analogous expressions not involving this approximation can be found and exploited, but only the planar case will be discussed in this report.

Substitution of equation (8) into equation (7) yields

$$A_n(\beta, \varphi) = \sum_{m=0}^{n-2} b_{nm} \left(\frac{2}{l}\right)^2 \frac{2}{\pi} \int_{x'=-l/2}^{l/2} \int_{y=y_1(x', \beta, \varphi)}^{y_2(x', \beta, \varphi)} t(x' + \beta y \cos \varphi, y) dy \left(\frac{x'}{l/2}\right)^m dx', \quad n \geq 2 \quad (9)$$

If it is understood that $t(x, y)$ is zero at points off the configuration rather than an analytic continuation of its form at points on the configuration, the integrations with respect to x' and y can both be taken from $-\infty$ to ∞ and the order of integration interchanged. In addition, with the substitution of $x = x' + \beta y \cos \varphi$, equation (9) becomes

$$A_n(\beta, \varphi) = \sum_{m=0}^{n-2} b_{nm} \left(\frac{2}{l}\right)^2 \frac{2}{\pi} \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} t(x, y) \left(\frac{x - \beta y \cos \varphi}{l/2}\right)^m dx dy, \quad n \geq 2 \quad (10)$$

The quantity in parentheses which is raised to the power m can be expanded by the binomial theorem into

$$\left(\frac{x - \beta y \cos \varphi}{l/2}\right)^m = \sum_{p=0}^m (-1)^p C_p^m \left(\frac{x}{l/2}\right)^{m-p} \left(\frac{\beta y \cos \varphi}{l/2}\right)^p \quad (11)$$

where

$$C_p^m = \frac{m!}{(m-p)!p!} \quad (12)$$

Then substituting equation (11) into equation (10) yields

$$A_n(\beta, \varphi) = \sum_{m=0}^{n-2} b_{nm} \left(\frac{2}{l}\right)^2 \frac{2}{\pi} \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \sum_{p=0}^m (-1)^p C_p^m \left(\frac{x}{l/2}\right)^{m-p} \left(\frac{\beta y \cos \varphi}{l/2}\right)^p t(x, y) dx dy$$

Upon interchange of orders of integration and summation, shortening of the interval of integration, and arbitrary grouping, this becomes

$$A_n(\beta, \varphi) = \sum_{m=0}^{n-2} b_{nm} \sum_{p=0}^m (-1)^p C_p^m \left(\frac{\beta \cos \varphi}{l/2} \right)^p \left(\frac{2}{l} \right)^{\frac{2}{\pi}} \int_{x=-l/2}^{l/2} \left[\int_{y=y_3(x)}^{y_4(x)} t(x, y) y^p dy \right] \left(\frac{x}{l/2} \right)^{m-p} dx \quad (13)$$

where $y_3(x)$ and $y_4(x)$ define the two edges of the configuration.

The quantity in brackets in equation (13) can be identified as the longitudinal distribution of the p th moment of area of the configuration. This indicates that the drag of the configuration can be expressed entirely in terms of moment distributions (including the area distribution which corresponds to $p = 0$).

The moment distributions can be defined as

$$M_p(x) = \int_{y=y_3(x)}^{y_4(x)} t(x, y) y^p dy \quad (14)$$

Substituting this in equation (13) yields

$$A_n(\beta, \varphi) = \sum_{m=0}^{n-2} b_{nm} \sum_{p=0}^m (-1)^p C_p^m \frac{2}{\pi} \left(\frac{2}{l} \right)^{p+2} \int_{-l/2}^{l/2} M_p(x) \left(\frac{x}{l/2} \right)^{m-p} dx \beta^p \cos^p \varphi \quad (15)$$

or interchanging the order of summations yields

$$A_n(\beta, \varphi) = \sum_{p=0}^{n-2} (-1)^p \beta^p \cos^p \varphi \left[\sum_{m=p}^{n-2} b_{nm} C_p^m \frac{2}{\pi} \left(\frac{2}{l} \right)^{p+2} \int_{-l/2}^{l/2} M_p(x) \left(\frac{x}{l/2} \right)^{m-p} dx \right] \quad (16)$$

as the desired expansion of the $A_n(\beta, \varphi)$'s in powers of β .

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It can be seen in equation (14) that if the configuration has spanwise symmetry, the odd moment distributions will be identically zero, and the terms of equation (16) resulting from odd values of p will be zero. With streamwise symmetry of the moment distributions in addition, the odd values of n and m would be eliminated.

In the process of substituting equation (16) into Jones' drag equation, it is convenient to define several new symbols.

Let

$$L_{np} = \sum_{m=p}^{n-2} b_{nm} c_p^m \frac{2}{\pi} \left(\frac{2}{l}\right)^{p+2} \int_{-l/2}^{l/2} M_p(x) \left(\frac{x}{l/2}\right)^{m-p} dx \quad (17)$$

so that

$$A_n(\beta, \phi) = \sum_{p=0}^{n-2} (-1)^p \beta^p \cos^p \phi L_{np} \quad (18)$$

Then $[A_n(\beta, \phi)]^2$ can be written as

$$[A_n(\beta, \phi)]^2 = \sum_{p_1=0}^{n-2} \sum_{p_2=0}^{n-2} (-1)^{p_1+p_2} L_{np_1} L_{np_2} \cos^{(p_1+p_2)}(\phi) \beta^{(p_1+p_2)}$$

or

$$[A_n(\beta, \phi)]^2 = \sum_{\substack{q=0 \\ q \text{ even}}}^{2n-4} \left[\left(L_{n, \frac{q}{2}} \right)^2 + 2 \sum_{p_1=q-n+2}^{\frac{q-2}{2}} L_{np_1} L_{n(q-p_1)} \right] \cos^q \phi \beta^q \quad (19)$$

where the odd values of q are omitted because the terms resulting from such values would not contribute to the drag.

In addition to L_{np} of equation (17), there are several other quantities depending upon L_{np} which are notational aids. Let

$$N_{nq} = \left(L_{n, \frac{q}{2}} \right)^2 \quad (20)$$

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and

$$I_{nq} = 2 \sum_{p_1=q-n+2}^{\frac{q-2}{2}} I_{np_1} I_{n(q-p_1)} \quad (21)$$

Then

$$[A_n(\beta, \varphi)]^2 = \sum_{\substack{q=0 \\ q \text{ even}}}^{2n-4} (N_{nq} + I_{nq}) \cos^q \varphi \beta^q \quad (22)$$

Substituting this in equation (3) yields

$$D = \frac{\rho V^2}{16} \int_0^{2\pi} \sum_{n=2}^{\infty} n \sum_{\substack{q=0 \\ q \text{ even}}}^{2n-4} (N_{nq} + I_{nq}) \cos^q \varphi \beta^q d\varphi$$

or, if it is assumed that the series may be integrated term-by-term, this becomes

$$D = \frac{\pi \rho V^2}{8} \sum_{n=2}^{\infty} n \sum_{\substack{q=0 \\ q \text{ even}}}^{2n-4} (N_{nq} + I_{nq}) J_q \beta^q \quad (23)$$

where

$$J_q = \frac{1}{2\pi} \int_0^{2\pi} \cos^q \varphi d\varphi = \frac{q!}{2^q \left[\left(\frac{q}{2} \right)! \right]^2}, \quad q \text{ even} \quad (24)$$

Interchanging the order of summations in equation (23) results in

$$D = \frac{\pi \rho V^2}{8} \sum_{\substack{q=0 \\ q \text{ even}}}^{\infty} J_q \beta^q \left[\sum_{n=\frac{q+4}{2}}^{\infty} n (N_{nq} + I_{nq}) \right] \quad (25)$$

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Let N_q be the quantity derived from N_{nq} by

$$N_q = \sum_{n=\frac{q+4}{2}}^{\infty} n N_{nq} \quad (26)$$

and define I_q as

$$I_q = \sum_{n=\frac{q+6}{2}}^{\infty} n I_{nq} \quad (27)$$

Then substituting these in equation (25) yields

$$D = \frac{\pi \rho V^2}{8} \sum_{\substack{q=0 \\ q \text{ even}}}^{\infty} J_q (N_q + I_q) \beta^q \quad (28)$$

as the desired expansion of the supersonic zero-lift drag formula in powers of β .

From the foregoing, it is seen that each N_q depends only upon the longitudinal distribution of the moment of area of order $q/2$, whereas each I_q depends on all moment distributions of order zero to $[(q/2) - 1]$. Thus, each N_q represents a contribution to the drag from the moment distribution of order $q/2$ alone, and each I_q represents a contribution resulting from the interaction of the first $[(q/2) - 1]$ moment distributions.

Although the question of convergence of the series of the foregoing analysis has not been investigated in detail, several observations and practical hints for calculation can be offered.

The values of each $A_n(\beta, \phi)$ obtained by the series-expansion method are identical to those obtained by the Mach plane method. Therefore, if it is assumed that the drag of a configuration can be calculated with sufficient accuracy by using the first N terms of the Mach plane method, it follows that the interchange in order of summations by which equation (25) is derived from equation (23) is valid for these terms.

In the Fourier series analysis of $\partial S(x', \beta, \phi) / \partial x'$ in the Mach plane method, it is evident that the higher harmonics will be suppressed if the

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smallest allowed value of λ is used rather than a large value. Consequently, the convergence of both the Mach plane and series-expansion methods is best when the smallest allowed value of λ is used in equations (1) and (17). In either method the number of terms required to obtain the major part of the drag can be held to a minimum, and the mathematical calculations thereby facilitated, by dividing the configuration under consideration into a short part and a long part. The quantities $A_n(\beta, \phi)$ of the complete configuration are the sums of the corresponding quantities of the separate short and long parts as given by the relation

$$A_n(\beta, \phi) = A_{Sn}(\beta, \phi) + A_{Ln}(\beta, \phi) \quad (29)$$

where the subscript S is used to denote short part and L long part. Then $[A_n(\beta, \phi)]^2$ is given by

$$[A_n(\beta, \phi)]^2 = [A_{Sn}(\beta, \phi)]^2 + 2A_{Sn}(\beta, \phi)A_{Ln}(\beta, \phi) + [A_{Ln}(\beta, \phi)]^2 \quad (30)$$

When this expression is substituted in equation (3) it is seen that the $[A_{Sn}(\beta, \phi)]^2$ will yield the drag of the short part alone, $[A_{Ln}(\beta, \phi)]^2$ the drag of the long part alone, and $2A_{Sn}(\beta, \phi)A_{Ln}(\beta, \phi)$ the interaction between the short and long parts. Then a smaller value of λ can be used to calculate the drag of the short part alone. Although no reduction in the value of λ is possible for the other two parts of the drag, the convergence is improved because of the absence of the high harmonic content of the short part from $A_{Ln}(\beta, \phi)$.

The number of terms to be included in the drag formula will depend upon the relative importance of accuracy and simplicity. In the search for low-drag design criteria, a very small number of terms might be appropriate. As an example of the meaning of this remark, it can be seen that only two terms need be considered to arrive at the supposition that the Sears-Haack area distribution is an optimum for given length and volume. Only one term is needed to conclude that the fineness ratio of a body should be as large as possible when the pressure drag alone is considered.

A Method for Reducing Drag

The general problem which will be considered in this section is that of designing a wing-body combination with low drag in a range of supersonic Mach numbers when certain basic parameters, such as total volume, are specified. In order to obtain definite answers, many additional parameters such as those involved in the specification of the wing plan form must be assigned arbitrarily. For example, the wing can be chosen

arbitrarily, and the optimum body shape for minimum zero-lift wave drag at a specified Mach number can be found by methods described in reference 1.

If the number of parameters affecting the drag is small, as is the case when the transonic area rule is valid, the general problem of drag minimization is greatly simplified. The series-expansion drag formula of the present analysis is also expressed in terms of a small number of parameters if the higher powers of β can be neglected or if the summations over n can be cut off at a small number. Because of the resulting simplification it has been found that the minimization procedure employed in reference 4 can be used to design an optimum wing-body combination with minimum drag at a Mach number of one and minimum drag rise at low supersonic speeds. As a first application of the foregoing analysis this procedure will be described and exploited. The validity of the result is subject to some question because the basic assumptions of the linear theory are violated at Mach numbers near one. However, the results are of interest in the absence of a method for applying a more exact theory.

For the ordinary case of spanwise symmetry, equation (28) can be written as

$$D = \frac{\pi \rho V^2}{8} \left[N_0 + \frac{1}{2} I_2 \beta^2 + \frac{3}{8} (N_4 + I_4) \beta^4 + \frac{5}{16} I_6 \beta^6 + \frac{35}{128} (N_8 + I_8) \beta^8 + O(\beta^{10}) \right] \quad (31)$$

In the speed and aspect-ratio range where β^2 and higher powers of β can be neglected, the drag depends only upon the area distribution, since this is the only feature of the geometry affecting N_0 . It can be assumed that the geometry will be such that the higher powers of β should be taken into account successively as the value of β is increased. Then in the speed range where powers of β greater than two can be neglected, the drag depends only on the area distribution and the second-moment-of-area distribution, since these two determine the value of I_2 . As successively higher powers of β are taken into account, correspondingly higher ordered moment distributions are involved.

The following procedure is proposed for reducing the wave drag of a wing-body configuration:

1. Minimize the drag at a Mach number of one by exclusive attention to the area distribution.
2. Minimize the drag at slightly higher Mach numbers by exclusive attention to the second-moment-of-area distribution excluding any changes which would alter the area distribution.

3. Continue to find the optimum higher ordered moment distributions successively without disturbing the lower distributions in their previously derived optimum forms.

This procedure leads to a unique set of moment distributions.

Substituting equation (20) into (26) with $q = 0$ yields

$$N_0 = \sum_{n=2}^{\infty} n (L_{n0})^2 \quad (32)$$

From equation (14) with $p = 0$, the volume of the configuration, V_0 , is identified as

$$V_0 = \int_{-l/2}^{l/2} M_0(x) dx \quad (33)$$

By the use of equation (17) with $n = 2$ and $p = 0$, the first term of equation (32) is found to be proportional to $(V_0/l^2)^2$. Then, since all the terms of equation (32) are positive, N_0 is a minimum for given values of V_0 and l if

$$L_{n0} = 0 \quad \text{for} \quad n \neq 2 \quad (34)$$

To satisfy equation (34) and similar equations which will occur, a rearrangement of equation (17) is needed. When the integral and sum of this equation are reversed to obtain

$$L_{np} = \frac{2}{\pi} \left(\frac{2}{l}\right)^{p+2} \int_{-l/2}^{l/2} M_p(x) \left[\sum_{m=p}^{n-2} b_{nm} C_p^m \left(\frac{x}{l/2}\right)^{m-p} \right] dx$$

the quantity in brackets can be identified as $\frac{1}{p!} \frac{d^{p+1}}{d\left(\frac{x}{l/2}\right)^{p+1}} \frac{\sin n\theta}{\sin \theta}$

with the aid of equations (5) and (12) so that

$$L_{np} = \frac{2}{\pi} \left(\frac{2}{l}\right)^{p+2} \int_{-l/2}^{l/2} \frac{M_p(x)}{p!} \frac{d^{p+1}}{d\left(\frac{x}{l/2}\right)^{p+1}} \frac{\sin n\theta}{\sin \theta} dx$$

Then after (p+1) partial integrations this becomes

$$I_{np} = \frac{2}{\pi} \frac{2}{l} \int_{-l/2}^{l/2} \left[\frac{d^{p+1}}{dx^{p+1}} \frac{M_p(x)}{p!} \right] \frac{\sin n\theta}{\sin \theta} dx \quad (35)$$

provided that

$$\left\{ \left[\frac{d^q}{d\left(\frac{x}{l/2}\right)^q} M_p(x) \right] \left[\frac{d^{(p-q)}}{d\left(\frac{x}{l/2}\right)^{p-q}} \frac{\sin n\theta}{\sin \theta} \right] \right\}_{x=l/2} - \left\{ \left[\frac{d^q}{d\left(\frac{x}{l/2}\right)^q} M_p(x) \right] \left[\frac{d^{(p-q)}}{d\left(\frac{x}{l/2}\right)^{p-q}} \frac{\sin n\theta}{\sin \theta} \right] \right\}_{x=-l/2} = 0 \quad (36)$$

for all integer values of q from zero to p . Equation (36) is satisfied if $M_p(x)$ and the first p derivatives of $M_p(x)$ are zero at $x = \pm l/2$.

It follows from equation (35) with $p = 0$ that equation (34) is satisfied by

$$\frac{dM_0}{dx} = L_{20} \sin(2\theta) = -2L_{20} \left(\frac{x}{l/2} \right) \sqrt{1 - \left(\frac{x}{l/2} \right)^2} \quad (37)$$

By integration and use of the fact that the configuration does not extend beyond $x = \pm l/2$, this becomes

$$M_0(x) = \frac{1}{3} l L_{20} \left[1 - \left(\frac{x}{l/2} \right)^2 \right]^{3/2} \quad (38)$$

For purposes of evaluating the drag, this area distribution can be put in the form

$$M_0(x) = M_0(0) \left[1 - \left(\frac{x}{l/2} \right)^2 \right]^{3/2} \quad (39)$$

where $M_0(0)$ is the maximum value of the distribution. Then

$$L_{20} = 3 \frac{M_0(0)}{l} \quad (40)$$

and

$$D = \frac{9\pi}{4} \rho V^2 \left[\frac{M_0(0)}{l} \right]^2 \quad (41)$$

is obtained as the drag of the optimum configuration in the speed range where β can be neglected. Equations (39) and (41) are in agreement with the results of references 4 and 5.

By substitution of equation (34) into equations (21) and (27), it is found that I_2 and I_4 will be zero for a configuration with the optimum area distribution and all other I_q 's will be independent of the area distribution. In that case equation (31) becomes

$$D = \frac{\pi \rho V^2}{8} \left\{ 18 \left[\frac{M_0(0)}{l} \right]^2 + \frac{3}{8} N_4 \beta^4 + \frac{5}{16} I_8 \beta^6 + \frac{35}{128} (N_8 + I_8) \beta^8 + O(\beta^{10}) \right\} \quad (42)$$

In minimizing N_4 , it can be assumed that the second-moment distribution of the body is negligible so that l can be replaced by l_2 , the wing length. More exactly, l_2 is the length of the projection of the wing on the body axis. Substituting equation (20) into (26) with $q = 4$, yields

$$N_4 = \sum_{n=4}^{\infty} n (L_{n2})^2 \quad (43)$$

The second-moment volume of the configuration can be defined as

$$V_2 = \int_{-l_2/2}^{l_2/2} M_2(x) dx \quad (44)$$

By the use of equation (17) with $p = 2$, the first term of equation (43) is found to be proportional to $[V_2/(l_2)^4]^2$. Then since all terms of equation (43) are positive, N_4 is a minimum for given values of V_2 and l_2 if

$$L_{n2} = 0 \quad \text{for} \quad n \neq 4 \quad (45)$$

It follows from equation (35) with $p = 2$ that this requirement is met by setting

$$\frac{d^3 M_2(x)}{dx^3} = L_{42} \sin(4\theta) \quad (46)$$

By triple integration and use of the sufficient requirement of equation (36) that $M_2(x)$ and the first two derivatives of $M_2(x)$ be zero at $x = \pm l_2/2$, this becomes

$$M_2(x) = \frac{1}{105} (l_2)^3 L_{42} \left[1 - \left(\frac{x}{l_2/2} \right)^2 \right]^{7/2} \quad (47)$$

or

$$M_2(x) = M_2(0) \left[1 - \left(\frac{x}{l_2/2} \right)^2 \right]^{7/2} \quad (48)$$

is obtained as the optimum second-moment distribution. Then

$$L_{42} = 105 \frac{M_2(0)}{(l_2)^3} \quad (49)$$

and by substitution of this in equations (42) and (43), the expression

$$D = \frac{\pi \rho V^2}{8} \left\{ 18 \left[\frac{M_0(0)}{l} \right]^2 + 16537.5 \left[\frac{M_2(0)}{(l_2)^3} \right]^2 \beta^4 \right\} \quad (50)$$

is obtained as the drag of a wing-body combination for which the distributions of area and second moment of area are optimum when powers of β greater than four are neglected.

The foregoing process can be continued indefinitely until a complete set of optimum moment distributions is obtained. The results are

$$M_p(x) = M_p(0) \left[1 - \left(\frac{x}{l_p/2} \right)^2 \right]^{p+(s/2)} \quad (51)$$

where

$$l_0 = l$$

$$l_p = l_2 \quad \text{for} \quad p \geq 2$$

and the drag of such a configuration is given by

$$D = \frac{\pi \rho V^2}{8} \sum_{\substack{p=0 \\ p \text{ even}}}^{\infty} \frac{[(2p)!]^3 (2p+3)^2 (2p+1)^2}{(p!)^6 (p+2) 2^{(2p-2)}} \left[\frac{M_p(0)}{(l_p)^{p+1}} \right]^2 \beta^{2p} \quad (52)$$

Since all the terms of equation (52) are positive, this drag and the corresponding drag coefficient must increase monotonically with increasing Mach number. Also since all the terms are positive it can be seen from the first two terms that the drag will be very large at

$$\beta^2 = \frac{M_0(0) l_2^3}{M_2(0) l} \quad (53)$$

Equation (53) can be used to estimate the upper limit of the Mach number range of applicability of the foregoing low supersonic technique for drag reduction.

It is interesting to note that a configuration designed according to equation (51) would have large drag at the higher Mach numbers as a consequence of eliminating the drag components due to interactions of the moment distributions. Although the drag due to each moment distribution alone must be positive, the interaction drags can be negative. The interaction drags which would be beneficial in a given Mach number increment were eliminated in the process of minimizing the drag at lower Mach numbers.

APPLICATIONS AND DISCUSSION

Design of Configuration for Low Drag

In the "ANALYSIS" section, it has been shown that the wave-drag formula can be expanded in a power series of the form

$$D = a_0 + a_2\beta^2 + a_4\beta^4 + \dots \quad (54)$$

where the constants a_0, a_2, a_4 , and so forth, are independent of Mach number and are determined only from the geometry of the configuration. In this section, the physical significance of these constants will be discussed and some examples presented of the practical means available for minimizing the constants in order to reduce the drag of wing-body combinations at low supersonic speeds as well as at Mach number one.

The details of the procedure for deriving the constants from the geometry using a planar approximation are contained in the "ANALYSIS" section and in Appendix B. Equations (17), (20), (21), (26), (27), and (28) indicate that, in general, a_0 depends only on the area distribution, a_2 depends upon the second-moment-of-area distribution as well as on the area distribution, and a_4 depends on the fourth-moment-of-area distribution in addition to the previous two distributions. Hence it is seen that the transonic area rule is valid in the speed range where β is small so that all terms except the first in equation (54) can be neglected. Furthermore, it is expected that as the Mach number is increased, starting from one, all except the first few terms should remain negligible in a range of low supersonic Mach numbers so that at these speeds the drag should depend only upon the area distribution and the second-moment-of-area distribution.

If the configuration area distribution is made an optimum for minimum drag at a Mach number of one by the use of the transonic area rule, the determination of a_2 and a_4 is simplified. In that case a_2 is zero and a_4 depends only on the second-moment-of-area distribution. Consequently, if the second-moment-of-area distribution can be varied without changing the area distribution, the drag at low supersonic speeds can be minimized with respect to such variations without increasing the sonic speed drag which depends only on the area distribution. In order to see that the second-moment distribution actually can be varied without changing the area distribution, definitions of these distributions are needed.

The area distribution is given by

$$M_0(x) = 2 \int_0^{Y(x)} t(x,y) dy \quad (55)$$

where $t(x,y)$ is the thickness distribution of the configuration including wing and body, and $Y(x)$ is the value of y at the edge of the plan form. In this definition spanwise symmetry of the configuration has been assumed.

The second-moment distribution can be approximated by

$$M_2(x) = 2 \int_{R(x)}^{Y(x)} y^2 t(x,y) dy \quad (56)$$

where $R(x)$ is the value of y at the wing-body juncture. Here the second-moment distribution of the body is neglected because of the small values of y at the body compared with those on the wing. Since the body moments are negligible, changes in the body shape will vary the configuration area distribution without altering the second-moment distribution. Conversely, the second-moment distribution can be altered while holding the area distribution fixed by varying the wing geometry and the body shape at the same time.

If the area and second-moment distributions are made optimum, the drag can still be varied by altering the magnitudes of these distributions, as can be seen in the drag formula for such a configuration given by

$$D = \frac{9}{4} \pi \rho V^2 \left[\frac{M_0(o)}{l} \right]^2 + \frac{3305}{16} \pi \rho V^2 \left[\frac{M_2(o)}{(l_2)^3} \right]^2 \beta^4 + O(\beta^6) \quad (57)$$

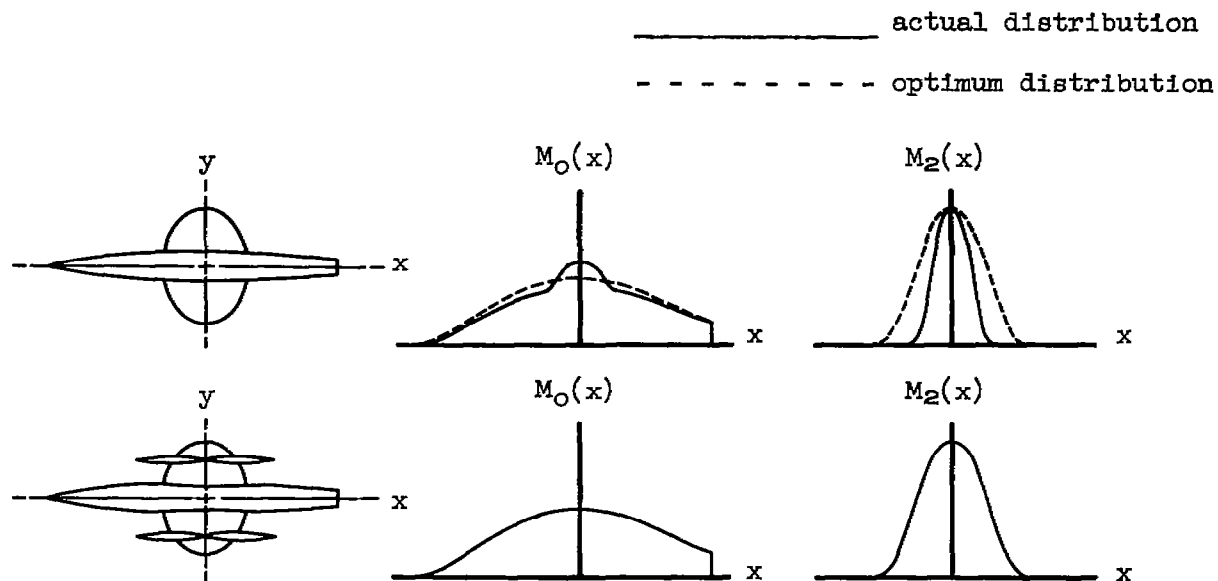
where $M_0(o)$ is the maximum value of the area distribution, l is the body length, $M_2(o)$ is the maximum value of the second-moment distribution, and l_2 is the wing length.

Neglecting powers of β greater than four in equation (57) provides an insight into the requirements for reducing the pressure drag at low supersonic speeds. The area-rule requirement that the ratio $M_0(o)/l$ be small indicates that the fineness ratio of the body should be large as previously noted. Examination of the quantity $M_2(o)/l_2^3$ leads to the conclusion that not only should the thickness ratio of the wing be small, but also the thickness ratio should taper to a minimum at the wing tips, and the ratio of effective streamwise length to span of the wing should be large. At higher Mach numbers, where the higher powers of β cannot be neglected, these conclusions would not apply.

The ratio of effective length to span of a wing can be increased in several different ways, for example, by extending the wing chord. However, large frictional drag penalties are usually associated with the increased surface area accompanying such changes. Another possible method of

increasing the effective-length-to-span ratio of the wing is by the addition of auxiliary bodies of revolution mounted on the wing. This method has the advantage of relatively small increased surface area and attendant friction drag. The bodies of revolution are particularly attractive in the case of wing-body configurations of relatively large wing span where application of the transonic area rule could be expected to produce drag penalties in the low supersonic speed region.

As an illustration of the application of the drag-reduction procedure, hereinafter referred to as the "moment-of-area rule" as distinguished from Whitcomb's area rule, consider the wing-body combination shown in the upper part of sketch (b). This configuration consists of a



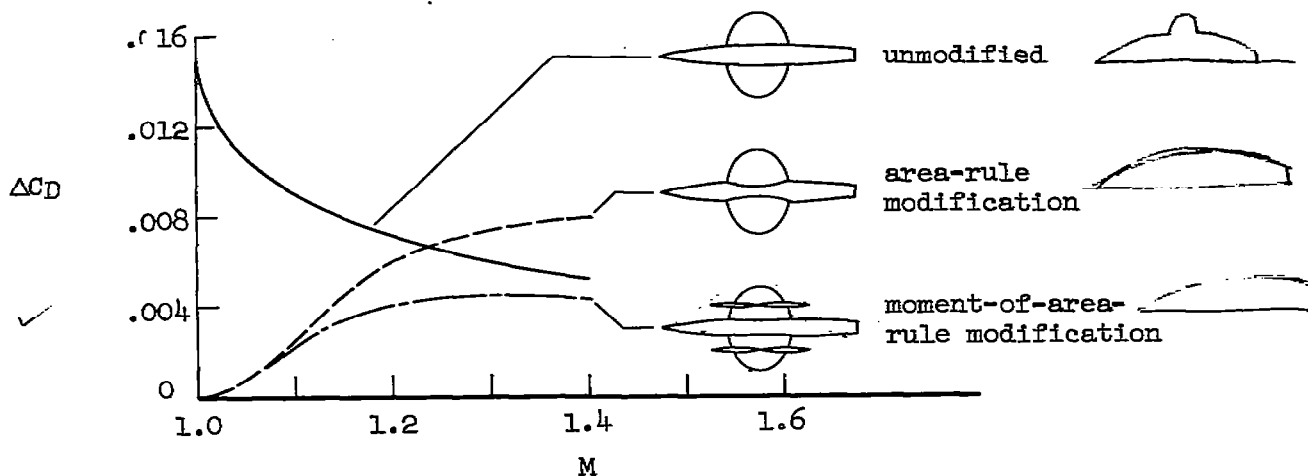
combination Sears-Haack-Kármán ogive body of fineness ratio 11, and an elliptic-plan-form wing of aspect ratio 2.0 with circular-arc sections and 5-percent maximum thickness ratio. The distributions of area and second moment of area for this basic configuration are also shown in the sketch. The shapes of these distribution curves are not conducive to low drag in that the area distribution has a bump at the location of the wing and the moment-of-area distribution is short and has steep slopes. With the total volume fixed, the optimum shapes of the distribution curves (as defined by eq. (51)) are shown by the dashed lines.² The desired distribution of the second moment of area can be obtained by utilizing auxiliary

²The optimum second-moment distribution is not a function of Mach number because it is derived essentially by minimizing the derivative $dC_D/d\beta^4$ evaluated at a Mach number of one.

bodies of revolution mounted on the wing as shown in the second part of sketch (b). The arbitrarily chosen spanwise location of the auxiliary bodies determines their size in that small bodies at an outboard position can produce the same second-moment distribution as larger bodies at an inboard position. It is evident that in order to prevent an increase in the maximum value of the second moment of area the auxiliary bodies must be waisted in the vicinity of the maximum thickness of the wing. The area distribution may be made optimum by reshaping the body to satisfy the requirements of the transonic area rule after the auxiliary bodies have been added.

In discussing the effects of modifications it is convenient to isolate portions of the drag which will not be affected by the modifications under consideration. Considering pressure drag only, the quantity of primary interest is the additional pressure drag caused by all additions to and alterations of the original body alone. The wing and auxiliary bodies are considered to be additions while the reshaping of the body is an alteration. Another reason for isolating this additional pressure drag (ΔC_D) is that the basic assumptions of the linear theory used to calculate ΔC_D for configurations with the transonic-area-rule modification may not be violated, although the assumptions are violated at Mach numbers near one for the body alone (see ref. 3).

The additional pressure drag as just defined is obtained by calculating the drag of a configuration consisting of the wing, the auxiliary bodies, and the body cutout. The body cutout is taken to be a negative area distribution located at the position of the body surface. The calculated values of ΔC_D for the unmodified configuration, and the configuration modified according to the moment-of-area rule are shown in sketch (c).



Sketch (c)

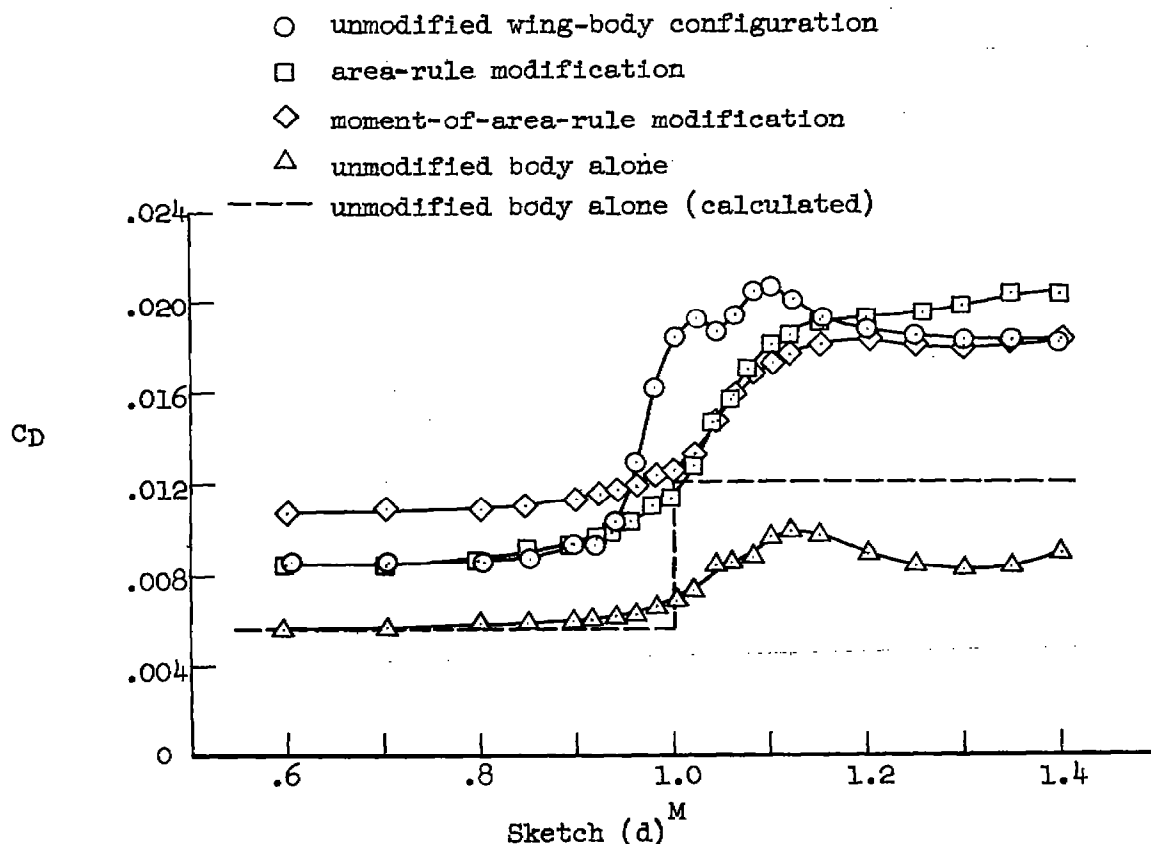
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For comparison, the calculated values of ΔC_D for a configuration modified according to the transonic area rule only are also shown. A description of the methods employed in the calculations appears in Appendixes B and C. The drag coefficient is based on the total wing area including the part of the wing hidden inside the body. It is apparent that the addition of the auxiliary bodies to make the second-moment-of-area distribution an optimum results in large theoretical drag reductions at low supersonic speeds. It is to be expected that the actual drag reductions will be somewhat less than those predicted because of the effect of friction drag not taken into account by the theory.

Experiment

In order to obtain an experimental check of the theoretical predictions, models of the configurations under consideration were constructed and tested in the Ames 2- by 2-foot transonic wind tunnel at a Reynolds number of 1.9 million based on the wing root chord.

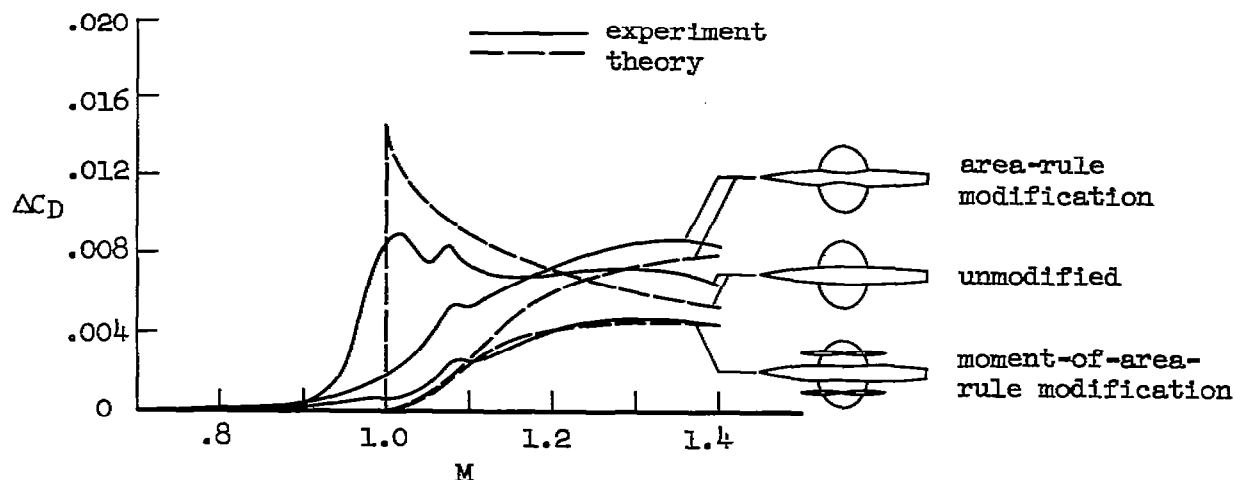
The experimentally measured values of the total drag coefficient at zero lift for the configurations are shown in sketch (d).

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These results confirm the predictions of the theory in that the drag of the configuration modified by the use of auxiliary bodies is nearly as low as that for the configuration modified by the area rule alone at a Mach number of 1.0 and is less at higher Mach numbers. The drag of the configuration with auxiliary bodies is greater at subsonic speeds than that of the other configurations because of the larger surface area.

The drag of the unmodified body alone is also shown in sketch (d) together with the predicted supersonic value which is plotted as an increment above the experimental value of drag at low speeds. The poor agreement is considered to result from a reduction of skin friction at supersonic speed due to an increase in the extent of the laminar flow since at supersonic speeds the extended regions of falling pressure are conducive to delay of transition to turbulent flow.

The incremental drag rises with the body drag excluded from the experimental and theoretical values show much better agreement as can be seen in sketch (e).



Sketch (e)

The ΔC_D values were estimated from the experimental results by subtracting the subsonic drag of the configuration as well as the drag rise of the original body alone from the total drag of the configuration. This operation can be expressed as:

$$\Delta C_D = C_{D_{\text{total}}} - \left(C_{D_{\text{total}}} \right)_{\text{at } M=0.8} - \left[C_{D_{\text{body alone}}} - \left(C_{D_{\text{body alone}}} \right)_{\text{at } M=0.8} \right]$$

or

$$\Delta C_D = \left(C_{D_{\text{total}}} - C_{D_{\text{body alone}}} \right) - \left(C_{D_{\text{total}}} - C_{D_{\text{body alone}}} \right)_{\text{at } M=0.8}$$

For the auxiliary bodies employed in the foregoing experimental investigation, the assumption that powers of β greater than four can be neglected in the drag formula is violated at Mach numbers above 1.1. Hence it is expected that modifications more effective in drag reduction at the higher Mach numbers can be found if this assumption is not used. The series-expansion method for evaluating the drag can be used to design auxiliary bodies which will minimize the pressure drag at a specified supersonic Mach number if the higher values of n are neglected rather than the higher powers of β . The result would correspond to the fuselage modification for minimum drag at a specified supersonic Mach number described in reference 1.

CONCLUDING REMARKS

A basic method for estimating the first-order deviations of the drag of wing-body combinations from the values predicted by the transonic area rule has been derived. In a planar approximation it has been found that at Mach numbers above one the zero-lift wave drag depends on the distributions of moments of area of the configuration about the vertical plane of symmetry parallel to the free-stream direction as well as on the area distribution. Thus the area rule can be supplemented by what might be termed a moment-of-area rule for extending to higher Mach numbers the drag reductions associated with the use of the area rule at a Mach number of one.

Just as is the case for the area rule where the longitudinal development of area must be smooth and gradual to minimize the drag, so also, in application of the moment of area rule, the longitudinal development of the moments of area must be smooth and gradual. It has been found that at low supersonic speeds the moment-of-area distributions of order higher than the second are of secondary importance. Significant drag reductions can be obtained at these speeds by mounting bodies of revolution on the wing for the purpose of improving the second-moment-of-area distribution. This point has been verified by an experiment performed in the Ames 2- by 2-foot transonic wind tunnel.

An alternative way to visualize the mechanism of drag reduction by this means is to regard the auxiliary wing-mounted bodies as local pressure-field cancellation devices in the same sense that Jones and Whitcomb employ the contoured principal body or fuselage to counteract

the pressure field for the entire wing. From this point of view, it should be expected that the auxiliary-body modification would be most applicable to configurations embodying wings of relatively large span where the area-rule effects would be limited because of the large distances of some of the wing elements from the fuselage.

The concept of introducing auxiliary bodies along the wing span to effect decreases in wave drag promises to find important application for aircraft intended to carry external stores. For such aircraft, the possibility exists of shaping the stores according to the moment of area rule so as to obtain drag reductions at transonic speeds with no friction penalty at lower speeds.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Oct. 19, 1954

APPENDIX A

TABLE OF SYMBOLS

a_n	coefficients of β^n in a power-series expansion of the drag
$A_n(\beta, \phi)$	coefficient of $\sin(n\theta)$ in a Fourier series expansion of $\partial S(x', \beta, \phi) / \partial x'$ (See eq. (2).)
b_{nm}	coefficient of $\left(\frac{x'}{l/2}\right)^m$ in a power-series expansion of $\frac{\partial \sin(n\theta) / \sin(\theta)}{\partial \left(\frac{x'}{l/2}\right)}$ (See eqs. (5) and (6).)
b	wing span
C_P^m	coefficient in a binomial expansion (See eqs. (11) and (12).)
D	zero-lift wave drag
$\left. \begin{matrix} I_q, I_{nq} \\ N_q, N_{nq} \end{matrix} \right\}$	quantities involved in evaluation of the drag (See eqs. (20), (21), (26), (27), and (28).)
J_q	(See eqs. (24) and (28).)
I_{np}	coefficient of $(\beta \cos \phi)^p$ in power-series expansion of $A_n(\beta, \phi)$ (See eqs. (17) and (18).)
l	body length
l_2	length of the longitudinal distribution of second moment of area (i.e., the length of the projection of the wing on the x axis)
M	free-stream Mach number
$M_0(x)$	longitudinal distribution of area of the configuration
$M_2(x)$	longitudinal distribution of moment of inertia about the xz plane (also called the second-moment-of-area distribution)

$M_p(o)$	maximum value of moment of area of order p
$M_p(x)$	longitudinal distribution of moment of area of order p (See eq. (14).)
m, n, p, q	dummy integers of summation
$R(x)$	value of y at the wing-body juncture
$S(x', \beta, \phi)$	area of the projection on the yz plane of the cross-sectional area intercepted on the configuration by the Mach plane (x', β, ϕ)
$t(x, y)$	thickness distribution of the configuration including wings and bodies
V	free-stream velocity
V_p	p th moment volume (See eqs. (33) and (44).)
x	Cartesian coordinate in the free-stream direction
y	Cartesian coordinate in the spanwise direction
z	Cartesian coordinate in the thickness direction
(x', β, ϕ)	Mach plane which intersects the x axis at x' and which has the projection of its normal on the yz plane at an angle ϕ to the y axis
$y_1(x', \beta, \phi)$ $y_2(x', \beta, \phi)$	values of y at the points of intersection of the configuration edges with the Mach plane (x', β, ϕ)
$y_3(x), y_4(x)$	values of y at the edges of the configuration
$Y(x)$	value of y at the edge of a configuration with spanwise symmetry
β	speed parameter equal to $\sqrt{M^2 - 1}$
θ	variable related to x' by $x' = \frac{l}{2} \cos \theta$
ρ	air density
ϕ	angle between the y axis and the projection on the yz plane of a normal to the Mach plane (x', β, ϕ)

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APPENDIX B

DIMENSIONLESS DRAG FORMULA

In order to take advantage of the decrease in the number of parameters resulting from similarity considerations, and to facilitate calculations, the quantities defined in the "ANALYSIS" section can be made dimensionless.

The moment-of-area distributions defined in equation (14) can be divided by a thickness t_0 and the half-span $b/2$ raised to the $(p + 1)$ th power to obtain dimensionless moment distributions defined as

$$\tilde{M}_p(x) = \frac{1}{t_0 \left(\frac{b}{2}\right)^{p+1}} M_p(x) = \frac{1}{t_0 \left(\frac{b}{2}\right)^{p+1}} \int_{y=-y_3(x)}^{y_4(x)} t(x,y) y^p dy \quad (B1)$$

Similarly, double moments occurring in equation (15) can be replaced by

$$\tilde{M}_{pk} = \frac{2/\pi}{\left(\frac{c}{2}\right)^{k+1}} \int_{-l/2}^{l/2} \tilde{M}_p(x) x^k dx \quad (B2)$$

where c is the length of the projection of the wing on the x axis.

With these definitions, a dimensionless version of the quantities I_{np} of equation (17) can be written as

$$\tilde{I}_{np} = \sum_{k=0}^{n-p-2} g_{npk} \tilde{M}_{pk} \left(\frac{c}{l}\right)^{k+2}, \quad n - p - 2 \geq 0 \quad (B3)$$

with the constants g_{npk} defined as

$$g_{npk} = g_{nkp} = \begin{bmatrix} \frac{n-p-k-2}{2} \frac{\left(\frac{n+p+k}{2}\right)!}{\left(\frac{n-p-k-2}{2}\right)!} \frac{2^k}{k!} \frac{2^p}{p!} & \text{for even values of } (n-p-k) \\ \text{zero otherwise} \end{bmatrix} \quad (B4)$$

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Dimensionless Fourier coefficients can be defined as

$$\tilde{A}_n = \frac{1}{2t_{ob}/c} A_n = \sum_{\substack{p=0 \\ p \text{ even}}}^{n-2} \tilde{I}_{np} \left(\frac{\beta b}{l} \cos \varphi \right)^p \quad (B5)$$

and \tilde{A}_n^2 would be written as

$$\tilde{A}_n^2 = \sum_{\substack{j=0 \\ j \text{ even}}}^{2n-4} \tilde{H}_{nj} \left(\frac{\beta b}{l} \cos \varphi \right)^j \quad (B6)$$

where

$$\tilde{H}_{nj} = \sum_{\substack{p=0 \\ p \text{ even}}}^j \tilde{I}_{np} \tilde{I}_{n(j-p)} \quad (B7)$$

p even since considering spanwise symmetry

Dimensionless drag components can be defined as

$$\tilde{D}_n = n \frac{1}{2\pi} \int_0^{2\pi} \tilde{A}_n^2 d\varphi \quad (B8)$$

so that

$$\tilde{D}_n = n \sum_{\substack{j=0 \\ j \text{ even}}}^{2n-4} \tilde{H}_{nj} J_j \left(\frac{\beta b}{l} \right)^j \quad (B9)$$

where

$$J_j = \frac{1}{2\pi} \int_0^{2\pi} \cos^j \varphi d\varphi = \frac{j!}{2^j \left[\left(\frac{j}{2} \right)! \right]^2}, \quad j \text{ even} \quad (B10)$$

Then the drag is given by

$$D = \pi q \left(\frac{t_{ob}}{c} \right)^2 \sum_{n=2}^{\infty} \tilde{D}_n \quad (B11)$$

where

$$q = \frac{1}{2} \rho V^2 \quad (B12)$$

The following is a suggested computing procedure:

1. Evaluate the dimensionless double moments \tilde{M}_{pk} for the configuration using equation (B2).
2. Choose a value of l equal to the length of the longest equivalent body of revolution and evaluate the \tilde{L}_{np} 's of equation (B3).
3. Evaluate the \tilde{H}_{nj} 's in equation (B7) and the \tilde{D}_n 's in equation (B9).

The dimensionless drag components $\tilde{D}_2, \tilde{D}_3, \tilde{D}_4$, and so forth, should be evaluated separately so that the convergence with respect to n can be watched and computing errors found more easily.

To determine which \tilde{M}_{pk} 's are needed for each value of n the sums can be written out proceeding in the opposite direction from that of the computing procedure. Thus

$$\left. \begin{aligned} \tilde{D}_2 &= 2\tilde{H}_{20} \\ \tilde{D}_3 &= 3\tilde{H}_{30} \\ \tilde{D}_4 &= 4 \left[\tilde{H}_{40} + \frac{1}{2} \tilde{H}_{42} \left(\frac{\beta b}{l} \right)^2 + \frac{3}{8} \tilde{H}_{44} \left(\frac{\beta b}{l} \right)^4 \right] \\ \tilde{D}_5 &= 5 \left[\tilde{H}_{50} + \frac{1}{2} \tilde{H}_{52} \left(\frac{\beta b}{l} \right)^2 + \frac{3}{8} \tilde{H}_{54} \left(\frac{\beta b}{l} \right)^4 \right] \\ \tilde{D}_6 &= 6 \left[\tilde{H}_{60} + \frac{1}{2} \tilde{H}_{62} \left(\frac{\beta b}{l} \right)^2 + \frac{3}{8} \tilde{H}_{64} \left(\frac{\beta b}{l} \right)^4 + \frac{5}{16} \tilde{H}_{66} \left(\frac{\beta b}{l} \right)^6 + \right. \\ &\quad \left. \frac{35}{128} \tilde{H}_{68} \left(\frac{\beta b}{l} \right)^8 \right] \\ \tilde{D}_7, \text{ etc.} \end{aligned} \right\} \quad (B13)$$

There will be $(n - 1)$ terms in \tilde{D}_n for even n and $(n - 2)$ terms for odd n .

Equation (B3) indicates that the quantities \tilde{L}_{np} with $p > n - 2$ are zero so that most of the terms in equations (B6) are zero. Consequently,

$$\left. \begin{aligned} \tilde{H}_{20} &= (\tilde{L}_{20})^2 \\ \tilde{H}_{30} &= (\tilde{L}_{30})^2 \\ \tilde{H}_{40} &= (\tilde{L}_{40})^2, & \tilde{H}_{42} &= 2\tilde{L}_{40} \tilde{L}_{42}, & \tilde{H}_{44} &= (\tilde{L}_{42})^2 \\ \tilde{H}_{50} &= (\tilde{L}_{50})^2, & \tilde{H}_{52} &= 2\tilde{L}_{50} \tilde{L}_{52}, & \tilde{H}_{54} &= (\tilde{L}_{52})^2 \\ \tilde{H}_{60} &= (\tilde{L}_{60})^2, & \tilde{H}_{62} &= 2\tilde{L}_{60} \tilde{L}_{62}, & \tilde{H}_{64} &= (\tilde{L}_{62})^2 + 2\tilde{L}_{60} \tilde{L}_{64} \\ \tilde{H}_{66} &= 2\tilde{L}_{62} \tilde{L}_{64}, & \tilde{H}_{68} &= (\tilde{L}_{64})^2 \\ \tilde{H}_{70}, \text{ etc.} \end{aligned} \right\} (B14)$$

From equations (B3) and (B4)

$$\left. \begin{aligned} \tilde{L}_{20} &= \tilde{M}_{00} \left(\frac{c}{l}\right)^2 \\ \tilde{L}_{30} &= 4\tilde{M}_{01} \left(\frac{c}{l}\right)^3 \\ \tilde{L}_{40} &= -2\tilde{M}_{00} \left(\frac{c}{l}\right)^2 + 12\tilde{M}_{02} \left(\frac{c}{l}\right)^4, & \tilde{L}_{42} &= 12\tilde{M}_{20} \left(\frac{c}{l}\right)^2 \\ \tilde{L}_{50} &= -12\tilde{M}_{01} \left(\frac{c}{l}\right)^3 + 32\tilde{M}_{03} \left(\frac{c}{l}\right)^5, & \tilde{L}_{52} &= 96\tilde{M}_{21} \left(\frac{c}{l}\right)^3 \\ \tilde{L}_{60} &= 3\tilde{M}_{00} \left(\frac{c}{l}\right)^2 - 48\tilde{M}_{02} \left(\frac{c}{l}\right)^4 + 80\tilde{M}_{04} \left(\frac{c}{l}\right)^6 \\ \tilde{L}_{62} &= -48\tilde{M}_{20} \left(\frac{c}{l}\right)^2 + 480\tilde{M}_{22} \left(\frac{c}{l}\right)^4 \\ \tilde{L}_{64} &= 80\tilde{M}_{40} \left(\frac{c}{l}\right)^2 \\ \tilde{L}_{70}, \text{ etc.} \end{aligned} \right\} (B15)$$

There will be $\frac{n-p}{2}$ terms in \tilde{L}_{np} for even values of n and $\frac{n-p-1}{2}$ terms for odd n .

Since only a small number of terms are involved in \tilde{D}_n for $n \leq 6$, these \tilde{D}_n 's can be conveniently written out in terms of \tilde{M}_{pk} 's as follows:

$$\begin{aligned}
 \tilde{D}_2 &= 2 \left[\tilde{M}_{00} \left(\frac{c}{l} \right)^2 \right]^2 \\
 \tilde{D}_3 &= 3 \left[4\tilde{M}_{01} \left(\frac{c}{l} \right)^3 \right]^2 \\
 \tilde{D}_4 &= 4 \left\{ \left[-2\tilde{M}_{00} \left(\frac{c}{l} \right)^2 + 12\tilde{M}_{02} \left(\frac{c}{l} \right)^4 \right]^2 + \left[-2\tilde{M}_{00} \left(\frac{c}{l} \right)^2 + \right. \right. \\
 &\quad \left. 12\tilde{M}_{02} \left(\frac{c}{l} \right)^4 \right] \left[12\tilde{M}_{20} \left(\frac{c}{l} \right)^2 \right] \left(\frac{\beta b}{l} \right)^2 + \frac{3}{8} \left[12\tilde{M}_{20} \left(\frac{c}{l} \right)^2 \right]^2 \left(\frac{\beta b}{l} \right)^4 \Big\} \\
 \tilde{D}_5 &= 5 \left\{ \left[-12\tilde{M}_{01} \left(\frac{c}{l} \right)^3 + 32\tilde{M}_{03} \left(\frac{c}{l} \right)^5 \right]^2 + \left[-12\tilde{M}_{01} \left(\frac{c}{l} \right)^3 + \right. \right. \\
 &\quad \left. 32\tilde{M}_{03} \left(\frac{c}{l} \right)^5 \right] \left[96\tilde{M}_{21} \left(\frac{c}{l} \right)^3 \right] \left(\frac{\beta b}{l} \right)^2 + \frac{3}{8} \left[96\tilde{M}_{21} \left(\frac{c}{l} \right)^3 \right]^2 \left(\frac{\beta b}{l} \right)^4 \Big\} \\
 \tilde{D}_6 &= 6 \left\{ \left[3\tilde{M}_{00} \left(\frac{c}{l} \right)^2 - 48\tilde{M}_{02} \left(\frac{c}{l} \right)^4 + 80\tilde{M}_{04} \left(\frac{c}{l} \right)^6 \right]^2 + \left[3\tilde{M}_{00} \left(\frac{c}{l} \right)^2 - \right. \right. \\
 &\quad \left. 48\tilde{M}_{02} \left(\frac{c}{l} \right)^4 + 80\tilde{M}_{04} \left(\frac{c}{l} \right)^6 \right] \left[-48\tilde{M}_{20} \left(\frac{c}{l} \right)^2 + 480\tilde{M}_{22} \left(\frac{c}{l} \right)^4 \right] \left(\frac{\beta b}{l} \right)^2 + \\
 &\quad \frac{3}{8} \left\{ 2 \left[3\tilde{M}_{00} \left(\frac{c}{l} \right)^2 - 48\tilde{M}_{02} \left(\frac{c}{l} \right)^4 + 80\tilde{M}_{04} \left(\frac{c}{l} \right)^6 \right] \left[80\tilde{M}_{40} \left(\frac{c}{l} \right)^2 \right] + \right. \\
 &\quad \left. \left[-48\tilde{M}_{20} \left(\frac{c}{l} \right)^2 + 480\tilde{M}_{22} \left(\frac{c}{l} \right)^4 \right]^2 \right\} \left(\frac{\beta b}{l} \right)^4 + \frac{5}{8} \left[-48\tilde{M}_{20} \left(\frac{c}{l} \right)^2 + \right. \\
 &\quad \left. 480\tilde{M}_{22} \left(\frac{c}{l} \right)^4 \right] \left[80\tilde{M}_{40} \left(\frac{c}{l} \right)^2 \right] \left(\frac{\beta b}{l} \right)^6 + \frac{35}{128} \left[80\tilde{M}_{40} \left(\frac{c}{l} \right)^2 \right]^2 \left(\frac{\beta b}{l} \right)^8 \Big\} \right\} \quad (B16)
 \end{aligned}$$

Because of the small differences between large numbers involved in the evaluation of \tilde{I}_{np} for large values of n , it is probably not feasible to calculate \tilde{D}_n for values of n as large as may be possible with the method of reference (1).

APPENDIX C

ALTERNATIVE DRAG FORMULA

In reference 4 it is shown that bodies of revolution exist which involve only a small number of values of n in the Fourier series expansion of the gradient of the body area distribution. Similarly, planar configurations exist which involve only a small number of values of n in equation (3) when l in equations (1) and (B3) is allowed to vary with polar angle ϕ and the speed parameter β . One such, a wing of elliptic plan form and circular-arc sections, was discussed in reference 6 and was shown to have minimum drag for given volume with elliptic plan form.

In this Appendix the drags of a series of wings of elliptic plan form are derived and the drag of an arbitrary planar configuration is expressed in terms of the elliptic-wing drags for the purpose of including the predominant effects of spanwise extension of the wing in a small number of terms of a series.

The double moments (\tilde{M}_{pk}) of an arbitrary planar configuration can be expressed in terms of double moments associated with an elliptic plan form which encloses the arbitrary plan form. Then the drag of the arbitrary configuration will be equal to the drag of the corresponding combination of elliptic-wing moments.

Equations (B3) and (B5) can be combined to obtain

$$\tilde{A}_n = \sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} g_{npk} \tilde{M}_{pk} \left(\frac{\beta b}{l} \cos \phi \right)^p \left(\frac{c}{l} \right)^{k+2} \quad (C1)$$

Taking l to be the length of the equivalent body of revolution for an elliptic plan form of span b_1 and maximum chord c_1 yields

$$l^2 = c_1^2 + (\beta b_1 \cos \phi)^2$$

or

$$\left(\frac{c_1}{l} \right)^2 = \frac{1}{1 + \left(\frac{\beta b_1}{c_1} \cos \phi \right)^2} = 1 - \left(\frac{\beta b_1}{l} \cos \phi \right)^2$$

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With the definition

$$\frac{c_1}{l} = \cos \alpha \quad (C2)$$

it is found that

$$\left(\frac{\beta b_1}{l} \cos \varphi \right)^p \left(\frac{c_1}{l} \right)^{k+2} = \sin^p \alpha \cos^{k+2} \alpha$$

so that

$$\tilde{A}_n = \sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} g_{npk} \tilde{M}'_{pk} \sin^p \alpha \cos^{k+2} \alpha \quad (C3)$$

where

$$\tilde{M}'_{pk} = \left(\frac{b}{b_1} \right)^p \left(\frac{c}{c_1} \right)^{k+2} \tilde{M}_{pk}$$

The dimensionless double moments \tilde{M}'_{pk} of the elliptic lens of reference 6 can be found by setting $\tilde{A}_n = 0$ for $n \neq 2$ and $\tilde{A}_2 = \cos^2 \alpha$ in equation (C3). Similarly, other wings of elliptic plan form can be defined by setting $\tilde{A}_n = 0$ for $n \neq n_1$ and $\tilde{A}_{n_1} = \cos^2 \alpha \sin^{p_1} \alpha$. Label this series of wings with the numbers n_1 and p_1 . The corresponding double moments $\tilde{M}_{n_1 p_1 k}$ of each such wing can be found for even values of n_1 and p_1 from the relation

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{n-2} \sum_{\substack{k=0 \\ k \text{ even}}}^{n-p-2} g_{npk} \tilde{M}_{n_1 p_1 k} \sin^{p_1} \alpha \cos^k \alpha = \delta_{nn_1} \sin^{p_1} \alpha \quad \text{for all values of } \alpha \text{ and } n \quad (C5)$$

where

$$\delta_{nn_1} = \begin{cases} 1 & \text{for } n = n_1 \\ \text{zero} & \text{otherwise} \end{cases} \quad (C6)$$

Equation (C5) can only be satisfied for values of $p_1 \leq n_1 - 2$ and it can be seen that the $\tilde{M}_{n_1 p_1 k}$'s are zero for $p < p_1$ or $k < n_1 - p - 2$. In the following discussion all integers have even values only.

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To obtain a drag formula, the \tilde{M}'_{pk} 's of the configuration are expressed as a combination of the $\tilde{M}_{n_1 p_1 p k}$'s of the form

$$\tilde{M}'_{pk} = \sum_{p_1=0}^p \sum_{n_1=p_1+2}^{p+k+2} \tilde{K}_{n_1 p_1} \tilde{M}_{n_1 p_1 p k} \quad (C7)$$

Substitution of this in equation (C3) yields

$$\tilde{A}_n = \cos^2 \alpha \sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} \sum_{p_1=0}^p \sum_{n_1=p_1+2}^{p+k+2} \delta_{npk} \tilde{K}_{n_1 p_1} \tilde{M}_{n_1 p_1 p k} \sin^{p_1} \alpha \cos^k \alpha$$

or interchange of the order of summations yields

$$\tilde{A}_n = \cos^2 \alpha \sum_{p_1=0}^{n-2} \sum_{n_1=p_1+2}^n \tilde{K}_{n_1 p_1} \sum_{p=p_1}^{n-2} \sum_{k=n_1-p-2}^{n-p-2} \delta_{npk} \tilde{M}_{n_1 p_1 p k} \sin^{p_1} \alpha \cos^k \alpha$$

By the use of equation (C5) this becomes

$$\tilde{A}_n = \cos^2 \alpha \sum_{p_1=0}^{n-2} \sum_{n_1=p_1+2}^n \tilde{K}_{n_1 p_1} \delta_{nn_1} \sin^{p_1} \alpha$$

Upon completion of the summation with respect to n_1 the expression

$$\tilde{A}_n = \cos^2 \alpha \sum_{p_1=0}^{n-2} \tilde{K}_{np_1} \sin^{p_1} \alpha \quad (C8)$$

is obtained and can be used in the place of equation (C3) in the drag formula if the \tilde{K}_{np_1} 's can be found. The \tilde{K}_{np_1} 's will be linear functions of the configuration \tilde{M}'_{pk} 's of the form

$$\tilde{K}_{np_1} = \sum_{p_2=0}^{p_1} \sum_{k_2=0}^{n-p_2-2} f_{np_1 p_2 k_2} \tilde{M}'_{p_2 k_2} \quad (C9)$$

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The quantities \tilde{K}_{np_1} can be found from the terms \tilde{M}_{pk} of the configuration if the $f_{n_1 p_1 p_2 k_2}$'s can be found. From equations (C3) and (C8) it is found that

$$\sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} g_{npk} \tilde{M}'_{pk} \sin^p \alpha \cos^k \alpha = \sum_{p_1=0}^{n-2} \tilde{K}_{np_1} \sin^{p_1} \alpha$$

for all α and all functions \tilde{M}'_{pk} . Substituting equation (C9) in the right side of the above equation yields

$$\begin{aligned} \sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} g_{npk} \tilde{M}'_{pk} \sin^p \alpha \cos^k \alpha &= \sum_{p_1=0}^{n-2} \sum_{p=0}^{p_1} \sum_{k=p_1-p}^{n-p-2} f_{np_1pk} \tilde{M}'_{pk} \sin^{p_1} \alpha \\ &= \sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} \tilde{M}'_{pk} \sum_{p_1=p}^{k+p} f_{np_1pk} \sin^{p_1} \alpha \end{aligned}$$

or

$$\sum_{p=0}^{n-2} \sum_{k=0}^{n-p-2} \tilde{M}'_{pk} \left(g_{npk} \sin^p \alpha \cos^k \alpha - \sum_{p_1=p}^{k+p} f_{np_1pk} \sin^{p_1} \alpha \right) = 0$$

for all α and all functions \tilde{M}'_{pk} . Thus

$$g_{npk} \sin^p \alpha \cos^k \alpha = \sum_{p_1=p}^{k+p} f_{np_1pk} \sin^{p_1} \alpha \quad \text{for all } \alpha$$

or expansion of $\cos^k \alpha$ by the binomial theorem yields

$$g_{npk} \sum_{p_1=p}^{k+p} (-1)^{\frac{p_1-p}{2}} \frac{k}{\frac{p_1-p}{2}} \sin^{p_1} \alpha = \sum_{p_1=p}^{k+p} f_{np_1pk} \sin^{p_1} \alpha$$

for all α so that

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$$f_{np_1pk} = (-1)^{\frac{p_1-p}{2}} c_{\frac{p_1-p}{2}}^2 \varepsilon_{npk} \quad (C10)$$

Substituting equations (12) and (B4) in this equation yields

$$f_{np_1pk} = (-1)^{\frac{n-p_1-k-2}{2}} \frac{\left(\frac{n+p+k}{2}\right)!}{\left(\frac{n-p-k-2}{2}\right)!} \frac{2^{p+k}}{p!k!} \frac{(k/2)!}{\left(\frac{k-p_1+p}{2}\right)! \left(\frac{p_1-p}{2}\right)!} \quad (C11)$$

for even values of the integers.

For odd values of n_1 and even p_1 the $\tilde{M}_{n_1p_1pk}$'s are defined by

$$\sum_{\substack{p=0 \\ p \text{ even}}}^{n-3} \sum_{k=1}^{n-p-2} \varepsilon_{npk} \tilde{M}_{n_1p_1pk} \sin^p \alpha \cos^{k-1} \alpha = \delta_{nn_1} \sin^{p_1} \alpha \quad (C12)$$

and by a similar process

$$\tilde{A}_n = \cos^3 \alpha \sum_{p=0}^{n-3} \tilde{K}_{np} \sin^p \alpha \quad (C13)$$

$$\tilde{K}_{np_1} = \sum_{\substack{p_2=0 \\ p_2 \text{ even}}}^{p_1} \sum_{\substack{k_2=p_1-p_2+1 \\ k_2 \text{ odd}}}^{n-p_2-2} f_{np_1p_2k_2} \tilde{M}'_{p_2k_2} \quad (C14)$$

and

$$f_{np_1pk} = (-1)^{\frac{p_1-p}{2}} c_{\frac{p_1-p}{2}}^2 \varepsilon_{npk} \quad (C15)$$

for odd values of n and k .

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Since the dimensionless drag components are given by

$$\tilde{D}_n = \frac{1}{2\pi} \int_0^{2\pi} n \tilde{A}_n^2 d\varphi$$

integrals of the type

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\left(\frac{\beta b_1}{c_1} \cos \varphi\right)^j}{\left[1 + \left(\frac{\beta b_1}{c_1} \cos \varphi\right)^2\right]^{\frac{j+4}{2}}} d\varphi, \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(\frac{\beta b_1}{c_1} \cos \varphi\right)^j}{\left[1 + \left(\frac{\beta b_1}{c_1} \cos \varphi\right)^2\right]^{\frac{j+6}{2}}} d\varphi$$

are encountered and have been evaluated by means of a method of residue integration and differentiation with respect to parameters.

In summary, the drag of an arbitrary closed planar configuration with spanwise symmetry can be evaluated by means of the following formulas:

$$D = \pi q \left(t_o \frac{b}{c}\right)^2 \sum_{n=2}^{\infty} \tilde{D}_n \quad (C16)$$

$$\tilde{D}_n = \begin{cases} n \sum_{\substack{j=0 \\ j \text{ even}}}^{2n-4} \tilde{H}_{nj} J_j I_j \left(\frac{\beta b_1}{c_1}\right) & \text{for even } n \\ n \sum_{\substack{j=0 \\ j \text{ even}}}^{2n-6} \tilde{H}_{nj} J_j Q_j \left(\frac{\beta b_1}{c_1}\right) & \text{for odd } n \end{cases} \quad (C17)$$

$$J_j = \frac{j!}{2^j \left[\left(\frac{j}{2}\right)!\right]^2} \quad \text{for even } j \quad (C18)$$

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$$I_j \left(\frac{\beta b_1}{c_1} \right) = \frac{\left(\frac{\beta b_1}{c_1} \right)^j \left[1 + \frac{1}{j+2} \left(\frac{\beta b_1}{c_1} \right)^2 \right]}{\left[1 + \left(\frac{\beta b_1}{c_1} \right)^2 \right]^{\frac{j+3}{2}}} \quad (C19)$$

$$Q_j \left(\frac{\beta b_1}{c_1} \right) = \frac{\left(\frac{\beta b_1}{c_1} \right)^j \left[1 + \frac{2}{j+2} \left(\frac{\beta b_1}{c_1} \right)^2 + \frac{3}{(j+2)(j+4)} \left(\frac{\beta b_1}{c_1} \right)^4 \right]}{\left[1 + \left(\frac{\beta b_1}{c_1} \right)^2 \right]^{\frac{j+5}{2}}} \quad (C20)$$

$$\tilde{H}_{nj} = \sum_{\substack{p=0 \\ p \text{ even}}}^j \tilde{K}_{np} \tilde{K}_{n(j-p)} \quad (C21)$$

$$\tilde{K}_{np} = \begin{bmatrix} \sum_{\substack{p_2=0 \\ p_2 \text{ even}}}^p \sum_{\substack{k_2=p-p_2 \\ k_2 \text{ even}}}^{n-p_2-2} f_{npp_2k_2} \tilde{M}_{p_2k_2} & \text{for even } n \\ \sum_{\substack{p_2=0 \\ p \text{ even}}}^p \sum_{\substack{k_2=p-p_2+1 \\ k_2 \text{ odd}}}^{n-p_2-2} f_{npp_2k_2} \tilde{M}_{p_2k_2} & \text{for odd } n \end{bmatrix} \quad (C22)$$

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$$f_{npp_2k_2} = \left[\begin{array}{l} (-1)^{\frac{n-p-k_2-2}{2}} \frac{\left(\frac{n+p_2+k_2}{2}\right)!}{\left(\frac{n-p_2-k_2-2}{2}\right)!} \frac{2^{(p_2+k_2)}}{p_2!k_2!} \frac{\left(\frac{k_2}{2}\right)!}{\left(\frac{k_2-p+p_2}{2}\right)! \left(\frac{p-p_2}{2}\right)!} \\ \text{for even } n \text{ and even } k_2 \\ \\ (-1)^{\frac{n-p-k_2-2}{2}} \frac{\left(\frac{n+p_2+k_2}{2}\right)!}{\left(\frac{n-p_2-k_2-2}{2}\right)!} \frac{2^{(p_2+k_2)}}{p_2!k_2!} \frac{\left(\frac{k_2-1}{2}\right)!}{\left(\frac{k_2-p+p_2-1}{2}\right)! \left(\frac{p-p_2}{2}\right)!} \\ \text{for odd } n \text{ and odd } k_2 \\ \\ \text{zero otherwise} \end{array} \right] \quad (C23)$$

$$\left. \begin{array}{l} \tilde{M}'_{p_2k_2} = \tilde{M}_{p_2k_2} \left(\frac{b}{b_1}\right)^{p_2} \left(\frac{c}{c_1}\right)^{k_2+2} \\ \\ \tilde{M}_{p_2k_2} = \frac{2}{\pi} \frac{1}{t_0 \left(\frac{b}{2}\right)^{p_2+1} \left(\frac{c}{2}\right)^{k_2+1}} \int_{-c/2}^{c/2} \int_{-y_3(x)}^{y_4(x)} t(x,y) y^{p_2} x^{k_2} dy dx \end{array} \right\} \quad (C24)$$

In equation (C24) $t(x,y)$ is the configuration thickness distribution, b_1 and c_1 are the span and maximum chord, respectively, of an ellipse which completely encloses the configuration, b is the wing span, and c is the streamwise length of the wing.

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The first five values of \tilde{D}_n are given by

$$\begin{aligned}
 \tilde{D}_2 &= 2(\tilde{M}'_{00})^2 I_0 \left(\frac{\beta b_1}{c_1} \right) \\
 \tilde{D}_3 &= 3(4\tilde{M}'_{01})^2 Q_0 \left(\frac{\beta b_1}{c_1} \right) \\
 \tilde{D}_4 &= 4 \left[(12\tilde{M}'_{02} - 2\tilde{M}'_{00})^2 I_0 \left(\frac{\beta b_1}{c_1} \right) + (12\tilde{M}'_{02} - 2\tilde{M}'_{00})(12\tilde{M}'_{20} - 12\tilde{M}'_{02}) I_2 \left(\frac{\beta b_1}{c_1} \right) + \frac{3}{8} (12\tilde{M}'_{20} - 12\tilde{M}'_{02})^2 I_4 \left(\frac{\beta b_1}{c_1} \right) \right] \\
 \tilde{D}_5 &= 5 \left[(32\tilde{M}'_{03} - 12\tilde{M}'_{01})^2 Q_0 \left(\frac{\beta b_1}{c_1} \right) + (32\tilde{M}'_{03} - 12\tilde{M}'_{01})(96\tilde{M}'_{21} - 32\tilde{M}'_{03}) Q_2 \left(\frac{\beta b_1}{c_1} \right) + \frac{3}{8} (96\tilde{M}'_{21} - 32\tilde{M}'_{03})^2 Q_4 \left(\frac{\beta b_1}{c_1} \right) \right] \\
 \tilde{D}_6 &= 6 \left\{ (80\tilde{M}'_{04} - 48\tilde{M}'_{02} + 3\tilde{M}'_{00})^2 I_0 \left(\frac{\beta b_1}{c_1} \right) + (80\tilde{M}'_{04} - 48\tilde{M}'_{02} + 3\tilde{M}'_{00})(480\tilde{M}'_{22} - 160\tilde{M}'_{04} + 48\tilde{M}'_{02} - 48\tilde{M}'_{20}) I_2 \left(\frac{\beta b_1}{c_1} \right) + \frac{3}{8} [2(80\tilde{M}'_{04} - 48\tilde{M}'_{02} + 3\tilde{M}'_{00})(80\tilde{M}'_{04} - 480\tilde{M}'_{22} + 80\tilde{M}'_{40}) + (480\tilde{M}'_{22} - 160\tilde{M}'_{04} + 48\tilde{M}'_{02} - 48\tilde{M}'_{20})^2] I_4 \left(\frac{\beta b_1}{c_1} \right) + \frac{5}{8} (480\tilde{M}'_{22} - 160\tilde{M}'_{04} + 48\tilde{M}'_{02} - 48\tilde{M}'_{20})(80\tilde{M}'_{04} - 480\tilde{M}'_{22} + 80\tilde{M}'_{40}) I_6 \left(\frac{\beta b_1}{c_1} \right) + \frac{35}{128} (80\tilde{M}'_{04} - 480\tilde{M}'_{22} + 80\tilde{M}'_{20})^2 I_8 \left(\frac{\beta b_1}{c_1} \right) \right\}
 \end{aligned} \tag{C25}$$

The convergence with respect to n is best when the smallest possible ellipse is used. The theoretical drag curves in sketch (e) were calculated by dividing the configuration into parts of short, long, and intermediate length so that smaller ellipses could be used for the shorter parts. The drags of these three parts and their interactions were calculated using values of n up to 6 in the formulas of this Appendix. The body moments in the region of the body cutouts were found to be important and were taken into account using a quasi-cylindrical approximation to find an equivalent planar system neglecting induced camber effects.

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